APPROXIMATING EXACT EXPECTED UTILITY VIA PORTFOLIO EFFICIENT FRONTIERS

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ABSTRACT

Expected utility theory is nowadays accepted as the standard for rational choice among risky assets. As pointed out by Markowitz (2014), the problem of how the maximum expected utility along the risk-return portfolio efficient frontiers approximates the exact maximum expected utility is still open. This paper analyzes how some popular risk-return models are able to approximate expected utility maximization, and provides some novel and robust results. Indeed, it also examines the ability of the whole risk-return efficient frontiers to approximate the exact one. Our empirical analysis is based on recent publicly available real-world data sets.

Keywords: Portfolio Optimization, Expected Utility, Multiobjective Optimization, Asset Management.

J.E.L. Classification: G10, G11, G12.

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1 Introduction

The start of Modern Portfolio Theory generally coincides with the publication of the Mean-Variance (MV) model of Markowitz (1952, 1959). Since its introduction, the MV model has both been praised and criticized: praises substantially concern its simplicity and its computational tractability, while criticisms concern its limited compatibility with the Expected Utility (EU) Theory. Indeed, it is often stated that the MV approach is compatible with that of EU only if the utility function is quadratic or if the assets returns follow a joint normal distribution, and that these two conditions are both restrictive. However, as Markowitz recently states, these criticisms fall into what he calls the "Great Confusion" between necessary and sufficient conditions for the use of MV analysis in practice. Indeed, if the condition on normal assets returns is not satisfied, MV can be still a good approximation of the EU criterion (Markowitz, 2012b; Markowitz and Blay, 2013; Markowitz, 2014).

Several studies aim at testing the goodness of such approximation. Levy and Markowitz (1979) compare exact EU with its Taylor expansion as a function of portfolio mean and variance for 149 mutual funds, showing their high correlation. Pulley (1981) investigates on MV approximations to EU for a wide class of utility functions, evaluating the goodness of these approximations by means of the ratio between the approximate and the exact EU. However, such a performance measure has been criticized by Kroll et al (1984), because it is not invariant with respect to positive affine transformations of the utility functions. For this reason, they compare the approximate maximum EU, obtained by maximizing EU among the MV efficient allocations, with the exact maximum EU by means of the performance measure of Pulley (1981) adjusted with the EU of the Equally-Weighted portfolio. Simaan (1993) evaluates the suboptimality of the approximate maximum EU, obtained by the MV approach, with respect to the exact one in terms of risk premium. He explores the problem considering the exponential utility function for several levels of the relative risk aversion coefficient, and using only simulated assets returns from a Pearson Type Three distribution. Hlawitschka (1994) compares different orders of the Taylor expansion for several utility functions, both in the case of convergent and of divergent Taylor series, evaluating the quality of the approximate EU with respect to the exact one by means of the Spearman rank correlation coefficient. More recently, Markowitz (2012a) studies six MV approximations to the portfolio geometric mean, namely the MV approximations to EU with the logarithmic utility. In another work, Markowitz (2012c) examines the portfolio geometric mean in terms of several risk measures via linear regression. However, as Markowitz (2014); Markowitz and Blay (2013) very recently pointed out, the problem of how the maximum EU along the risk-return efficient frontiers approximates the exact EU is still open.

Our paper contributes to this strand of research by extending the analysis to other popular risk measures and using on some new publicly available real-world data sets. In particular our extensive empirical analysis is performed by means of a rolling time windows scheme and the goodness of approximation is evaluated both with respect to optimal values and to optimal solutions. Furthermore, we also examine the ability of the whole risk-return efficient frontier to approximate the exact one.

The structure of this paper is as follows. In Section 2 we introduce the global objective of maximizing expected utility and we present a Generalized Risk-Gain problem based on the disaggregation of the global objective in two partial objectives, where the Generalized Risk depends on all the central moments of order higher than 1. Furthermore, we describe some famous risk-return models for which we examine their ability to approximate expected utility maximization. We also provide the description of the methodologies used to evaluate such approximation. In Section 3 we illustrate the data sets considered, and we discuss the main results of the empirical analysis both in the single and in the rolling time windows scheme. Conclusions are drawn in Section 4, where we also describe some issues left for future research.

2 Exact vs. Approximate Expected Utility

In Expected Utility Theory an investor evaluates a random portfolio wealth W_P with respect to her preferences, represented by a utility function u. She will choose the portfolio with the maximum expected utility. Formally, we have

$$\max_{x\in C} E[u(W_P(x))] \tag{1}$$

where x is the vector of portfolio weights and C is a set of feasible portfolios. In this paper we consider the long-only portfolios satisfying the budget constraint, $x \in \Delta = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, x_i \geq 0, \quad i = 1, \ldots, n\}$.

We can introduce the following generalized risk measure

$$\Phi(W_P(x)) = u(E[W_P(x)]) - E[u(W_P(x))$$

which, by *Jensen's inequality* (Jensen, 1906), is not negative for a risk-averse investor, and it is null only when the portfolio is risk-free or when the utility function is linear. Then, the global objective of maximizing expected utility can be disaggregated in two partial objectives: the maximization of $u(E[W_P(x)])$ and the minimization of $\Phi(W_P(x))$. Therefore, we consider the following bi-objective optimization problem

$$\begin{array}{ll} \min & \Phi(W_P(x)) \\ \max & u(E[W_P(x)]) \\ s.t. & \\ & x \in \Delta \end{array} \tag{2}$$

which is the standard representation of the Risk-Gain analysis, and we denote by EF the set of efficient portfolios obtained by solving Problem (2). It is straightforward to see that Problem (1) is equivalent to solve

$$\max_{x\in EF} E[u(W_P(x))] \tag{3}$$

Note that if we consider the Taylor expansion of the Expected Utility (EU) around $\overline{W} = E[W_P(x)]$, then

$$\Phi(W_P(x)) = \sum_{k=2}^{+\infty} \frac{1}{k!} u^{(k)}(\bar{W}) E[(W_P(x) - \overline{W})^k] \, ,$$

hence, the set of efficient portfolios EF can be obtained by maximizing the expected portfolio wealth $E[W_P(x)]$, i.e., the raw moment of order 1, and by minimizing the generalized risk measure $\Phi(W_P(x))$, which depends on all the central moments of order higher than 1.

In this work, we examine how some famous risk-return models are able to approximate expected utility maximization. More precisely, we first detect the set of efficient portfolios $\mathcal{EF}$, obtained by replacing the generalized risk measure $\Phi(W_P(x))$ with some symmetric (Variance, MAD) and asymmetric (CVaR, MinMax) risk measures. Then, we solve Problem (3) with $x \in \mathcal{EF}$, and we test the quality of this approximation with respect to the exact one.

In the following sections we first describe the optimization problem for the exact expected utility maximization (see Section 2.1), and then for the risk-return models analyzed (see Section 2.2).

Let us now introduce some notations. Let $T + 1$ be the length of the in-sample period used to estimate the inputs for the models. We use p_{it} to denote the price of the i-th asset at time t, with $t = 1, ..., T$; $r_{it} = \frac{p_{it} - p_{i(t-1)}}{p_{i(t-1)}}$ the i-th asset return at time t; x is the vector whose components x_i are the fractions of a given capital invested in asset i in the portfolio we are selecting. We assume that n assets are available in a market and, adopting linear returns, we have that $R_t(x) = \sum_{i=1}^{n} x_i r_{it}$ is the portfolio return at time t. We denote by μ_i the expected return of asset i, and by σ_{ij} the covariance of returns of asset i and asset j for $i, j = 1, \ldots, n$. Furthermore, we indicate the portfolio wealth at time t by $W_t(x) = W_{t-1}(1 + R_t(x))$.

2.1 The optimization problem for EU maximization

As mentioned above, the problem of choice under uncertainty conditions based on the EU criterion consists in maximizing the EU of the wealth $W_P(x)$, namely

$$\begin{aligned} E[u(W_P(x))] &= E[u\left(w_0(1 + R_P(x))\right)] = & (4) \\ &= E\left[u\left(w_0\left(1 + \sum_{i=1}^{n} x_i r_i\right)\right)\right] & (5) \end{aligned}$$

where w_0 is the initial wealth. As in Markowitz and Blay (2013), in the discrete case we can write

$$E[u(W_P(x))] = \frac{1}{T} \sum_{t=1}^{T} u\left(w_0\left(1 + \sum_{i=1}^{n} x_i r_{it}\right)\right) \qquad (6)$$

	Utility function
Logarithmic	$u(W) = \ln(W)$
Power	$u(W) = W^a$ with $a = 0.01, 0.1, 0.5, 0.9$
Exponential	$u(W) = -e^{-bW}$ with $b = 0.5, 1, 3, 5, 10$

Table 1: List of the utility functions considered.

Therefore, the expected utility maximization problem is

$$\begin{cases} \max_x & \frac{1}{T}\sum_{t=1}^{T} u\left(w_0\left(1+\sum_{i=1}^{n} x_i r_{it}\right)\right) \\ \text{s.t.} & \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0 \qquad\qquad i = 1,\ldots,n \end{cases} \tag{7}$$

where in our analysis $w_0 = 1$, and u is an increasing and concave function ($u' > 0$ and $u'' < 0$), representing the preference of a risk-averse investor. As specified in Section 3, all models are implemented in Matlab. Specifically, the convex optimization problem (7) is solved by using the built-in function *fmincon*, which allows for choosing among different optimization algorithms. In order to select the best algorithm in terms of efficiency we extensively test the sequential quadratic programming (sqp), the interior-point, and the active-set algorithms, available on Matlab. To check the stability of the optimal solutions with respect to the starting point, we solve Problem (7) by considering random starting points uniformly distributed over the simplex, generated by an algorithm provided by Rubinstein (1982). Thus, comparing the three algorithms in terms of stability and efficiency, we decided to use the *sqp* algorithm.

Note that in Problem (7) we obtain the same optimal solution if we replace the utility function u with any affine transformation of u.

2.1.1 Some Utility Functions

In our empirical analysis we consider the utility functions listed in Table 1, and used in Hlawitschka (1994); Kroll et al (1984); Levy and Markowitz (1979); Markowitz (2012b, 2014); Pulley (1981). As is well known, such utility functions are coherent with the non-satiation property and with the assumption of risk-averse operators, i.e., $u \in U_2$ ($u' \geq 0$, $u'' \leq 0$).

Remark 1 (Relation between utility of return and of wealth) *As shown, e.g., in Markowitz (2014) for the logarithmic and power utility functions it is equivalent to directly maximize the portfolio return or to maximize the portfolio wealth. Indeed,* *for the logarithmic utility function we have*

$$\max \ln(W_t(x)) = \max \ln(W_{t-1}(1 + R_t(x))) = \ln(W_{t-1}) + \max \ln(1 + R_t(x)).$$

Similarly, for the power utility function we can write

$$\max W_t^a(x) = \max W_{t-1}^a(1 + R_t(x))^a = W_{t-1}^a \max(1 + R_t(x))^a.$$

Note that when we select $R_t(x)$, W_{t-1} is given, since it concerns the previous period. Therefore, it can be treated as a constant.

2.2 Risk-return portfolio models

In this section we present the risk-return portfolio models analyzed, providing a formulation for all of them. As mentioned above, adopting a two-stage approach we test how these models approximate the expected utility. More precisely, we first compute the efficient frontiers $\mathcal{EF}$ for a given risk-return model. Then, we select the portfolio on $\mathcal{EF}$ which maximizes the expected utility, namely:

$$\tilde{x}^* = \arg\max_{x \in \mathcal{EF}} E[u(W_P(x))] \tag{8}$$

where $\mathcal{EF} \subseteq \Delta = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x_i \geq 0, \quad i = 1, \ldots, n\}$ is the set of efficient allocations in terms of risk-return analysis. Finally, we compare the approximate Expected Utility $E[u(W_P(\tilde{x}^*))]$ with respect to the exact one, $E[u(W_P(x^*))]$, where

$$x^* = \arg\max_{x \in \Delta} E[u(W_P(x))]\,, \tag{9}$$

and clearly $E[u(W_P(\tilde{x}^*))] \leq E[u(W_P(x^*))]$.

As described in the following sections, the risk-return portfolio models analyzed can be formulated by linear and quadratic programming problems solved using the GUROBI/CPLEX toolbox Gurobi Optimization (2015) called from Matlab.

2.2.1 The Markowitz portfolios

The classical Mean-Variance (MV) portfolio optimization model aims at determining the fractions x_i of a given capital to be invested in each asset i belonging to a predetermined investment universe so as to minimize the risk of the return of the whole portfolio, identified with its variance, while restricting the expected return of the portfolio to attain a specified target level.

More precisely, using the notation introduced above and denoting by η the required level of the portfolio expected return, the MV model is:

$$\begin{aligned}
\min \quad & \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} x_i x_j \\
s.t. \quad & \\
& \sum_{i=1}^n \mu_i x_i = \eta \\
& \sum_{i=1}^n x_i = 1 \\
& x_i \geq 0 \qquad\qquad i = 1, \ldots, n
\end{aligned} \tag{10}$$

This is a convex quadratic programming problem which can be solved by a number of efficient algorithms with a moderate computational effort even for large instances.

We denote by $\sigma_P^2(\eta)$ the optimal value of (10) as a function of η. Let η_{min} denote the value of $\sum_{i=1}^n \mu_i x_i$ at an optimal solution of the problem obtained by deleting the first constraint in (10), and let $\eta_{max} = \max\{\mu_1, \ldots, \mu_n\}$. Then the graph of $\sigma_P^2(\eta)$ on the interval $[\eta_{min}, \eta_{max}]$ coincides with the set of all efficient portfolios, and is usually approximated by solving (10) for several (equally spaced) values of η in $[\eta_{min}, \eta_{max}]$.

2.2.2 The Semi-MAD portfolios

Another symmetric risk measure that we take into account in our analysis is the downside Mean Semi-Absolute Deviation (Semi-MAD):

$$SMAD(x) = E[\max(0, \sum_{i=1}^{n} (\mu_i - r_{it}) x_i)], \tag{11}$$

This is a concise version of the more famous Mean Absolute Deviation (MAD) risk measure, which is defined as the expected value of the absolute deviation of the portfolio return from its mean (Konno and Yamazaki, 1991). Indeed, Speranza (1993) showed that Semi-MAD leads to a portfolio selection model that is equivalent to the MAD model, but with half the number of constraints.

We thus consider the following Semi-MAD (SMAD) model

$$\begin{array}{ll} \min & \frac{1}{T}\sum_{t=1}^{T} \max(0, \sum_{i=1}^{n} (\mu_i - r_{it}) x_i) \\ s.t. & \\ & \sum\limits_{i=1}^{n} \mu_i x_i = \eta \\ & \sum\limits_{i=1}^{n} x_i = 1 \\ & x_i \geq 0 \qquad i = 1, \ldots, n \end{array} \tag{12}$$

As in Speranza (1993) we linearize Problem (12) by introducing T auxiliary variables z_t defined as the deviation of the portfolio returns R_t from its mean when $R_t < \sum_{i=1}^n \mu_i x_i$, and 0 otherwise. Therefore, we substitute $\max(0, \sum_{i=1}^n (\mu_i - r_{it}) x_i) = z_t$ by adding the following constraints: $z_t \geq 0$ and $z_t \geq \sum_{i=1}^n (\mu_i - r_{it}) x_i$. Hence, the SMAD model can be re-written as the following Linear Programming (LP) problem:

$$\begin{array}{lll} \min & \dfrac{1}{T}\sum\limits_{t=1}^{T} z_t & \\ s.t. & & \\ & z_t \geq \sum_{i=1}^{n} (\mu_i - r_{it}) x_i, & t = 1, \ldots, T \\ & z_t \geq 0, & t = 1, \ldots, T \\ & \sum\limits_{i=1}^{n} \mu_i x_i = \eta & \\ & \sum\limits_{i=1}^{n} x_i = 1 & \\ & x_i \geq 0 & i = 1, \ldots, n \end{array} \tag{13}$$

This problem has $n+T$ variables and $T+2$ constraints.

2.2.3 The CVaR portfolios

The Conditional Value-at-Risk (CVaR) model is a risk-return model like the previous ones, but with CVaR at a specified confidence level ε ($CVaR_\varepsilon$) as risk measure, namely the average of losses in the worst $100\varepsilon\%$ of the cases (Acerbi and Tasche, 2002). The formal definition of CVaR is:

$$CVaR_\varepsilon(x) = -\frac{1}{\varepsilon}\int_0^\varepsilon Q_{R_P(x)}(\alpha)d\alpha\,,$$

where $Q_{R_P(x)}(\alpha)$ is the quantile function of the portfolio return $R_P(x)$. As usual, losses are defined as negative outcomes, and we set ε equal to 0.05. As described in Rockafellar and Uryasev (2000), in the discrete case we have

$$\min_x CVaR_\varepsilon(x) = \min_{(x,\zeta)} \zeta + \frac{1}{\varepsilon}\sum_{t=1}^T p_t[\sum_{i=1}^n -r_{it}x_i - \zeta]^+, \tag{14}$$

where $\zeta \in \mathbb{R}$, $[b]^+ = \max\{0, b\}$, and p_t is the probability of the historical scenario of the portfolio losses $l_t = \sum_{i=1}^n -r_{it}x_i$. We assume that all scenarios are equally likely, i.e., $p_t = 1/T$. Furthermore, to linearize the objective function, we introduce T auxiliary variables d_t that are defined as the deviations of the portfolio losses l_t from ζ when $l_t > \zeta$, and 0 otherwise. Thus, the CVaR model can be formulated as the following LP problem

$$\begin{array}{lll}
\min & \zeta + \frac{1}{\varepsilon}\frac{1}{T}\sum_{t=1}^T d_t & \\
s.t. & & \\
& \sum_{i=1}^n -r_{it}x_i - d_t - \zeta \le 0, & t = 1,\dots,T \\
& d_t \ge 0, & t = 1,\dots,T \\
& \sum_{i=1}^n \mu_i x_i = \eta & \\
& \sum_{i=1}^n x_i = 1 & \\
& x_i \ge 0 & i = 1,\dots,n \\
& \zeta \in \mathbb{R} &
\end{array} \tag{15}$$

Problem (15) has $n+T+1$ variables and $T+2$ constraints.

2.2.4 The MinMax portfolios

The *MinMax* model (Young, 1998), like the CVaR model, is based on an asymmetric risk measure. Note that the MinMax model is a special case of the CVaR model when the confidence level $\varepsilon \to 0$. In other words, this model aims at finding the

long-only portfolio that minimizes the maximum loss and that satisfies the usual constraints on the budget and on the required target level of the portfolio expected return. Alternatively, one can also solve the problem of maximizing the minimum portfolio return.
Let R_P^{min} denote the minimum return of a portfolio, $R_P^{min} = \min_{1\leq t\leq T} \sum_{i=1}^n x_i r_{it}$, the MinMax model can be formulated as

$$\begin{array}{ll} \max\limits_{x} & \min\limits_{1\leq t\leq T} \sum\limits_{i=1}^n x_i r_{it} \\ st & \\ & \sum\limits_{i=1}^n \mu_i x_i = \eta \\ & \sum\limits_{i=1}^n x_i = 1 \\ & x_i \geq 0 \qquad\qquad i = 1,\ldots,n \end{array} \tag{16}$$

Problem (16) can be linearized in the usual manner by adding an auxiliary variable d as follows

$$\begin{array}{ll} \max\limits_{x,d} & d \\ st & \\ & d \leq \sum\limits_{i=1}^n x_i r_{it} \qquad t = 1,\ldots,T \\ & \sum\limits_{i=1}^n \mu_i x_i = \eta \\ & \sum\limits_{i=1}^n x_i = 1 \\ & x_i \geq 0 \qquad\qquad i = 1,\ldots,n \end{array} \tag{17}$$

2.3 Indexes for evaluating the quality of the approximation

As observed in Section 2.2, we have $E[u(W_P(\tilde{x}^*))] \leq E[u(W_P(x^*))]$. However, the quality of the approximation cannot be measured by an easy difference $\Delta EU = E[u(W_P(x^*))] - E[u(W_P(\tilde{x}^*))]$, since ΔEU is not invariant to positive affine transformations. Accordingly, as in Kroll et al (1984), we consider the Equally-Weighted (EW) portfolio (i.e., $x_i^{EW} = 1/n$ for all $i = 1,\ldots,n$) as a reference portfolio, and we evaluate the quality of the approximation by the following index

$$I_{appr} = \frac{E[u(W_P(\tilde{x}^*))] - E[u(W_P(x^{EW}))]}{E[u(W_P(x^*))] - E[u(W_P(x^{EW}))]}, \tag{18}$$

that we call *Approximation Index*, where $E[u(W_P(x^{EW}))]$ represents the expected utility of the EW portfolio.

Additionally, we use another index for measuring the accuracy of the approximation, i.e., the norm of the difference between the solution x^*, obtained by the exact expected utility maximization (9), and $\tilde{x}^*$ the optimal solution to (8). More in detail, we compute

$$I_{dist} = \frac{1}{Q} \sum_{k=1}^{Q} \|\tilde{\boldsymbol{x}}_k^* - \boldsymbol{x}_k^*\|$$

where Q is the number of rebalances realized for the rolling time window analysis (see Section 3.2.2).

Finally, we also test how $\mathcal{EF}$ approximates EF. More precisely, in the $\Phi - UE$ plane (see Section 2) for a fixed target return level η, we examine the distance between the risk Φ of the portfolio belonging to EF and that of the portfolio belonging to $\mathcal{EF}$. For this purpose, we use the index defined, for instance, in Cesarone et al (2013), named *Average Percentage Loss* (APL):

$$APL = \sum_{j=1}^{100} \frac{\Phi(\tilde{\boldsymbol{x}}_j^*) - \Phi(\boldsymbol{x}_j^*)}{\Phi(\boldsymbol{x}_j^*)}, \tag{19}$$

where $\Phi(\boldsymbol{x}_j^*)$ is the generalized risk corresponding to the optimal solution $\boldsymbol{x}_j^*$ to Problem (2) and $\Phi(\tilde{\boldsymbol{x}}_j^*)$ is the generalized risk corresponding to the optimal solution $\tilde{\boldsymbol{x}}_j^*$ to the risk-return models (described in Section 2.2) for a fixed target return η_j. The returns η_j (with $j = 1, \ldots, 100$) are equally distributed in the interval $[\eta_1, \eta_{max}]$, where η_1 is the maximum value of η_{min} obtained by all models considered. Furthermore, we compute the average of APL on all the in-sample periods of the rolling time window analysis.

3 Empirical Analysis

In this section we present an extensive empirical analysis, performed on five real-world data sets that are described in the following section.

All models have been implemented in Matlab 8.5 on a workstation with Intel Core CPU (i7-6700, 3.4 GHz, 16 Gb RAM) under MS Windows 10.

3.1 Data sets

We provide here some details about the five real-world data sets, consisting of daily prices obtained from Thomson Reuters Datastream:

- DJIA, containing 28 assets of the Dow Jones Industrial Average Market Index (USA) from 16/02/1990 to 07/04/2016 (daily frequency);
- HSI, containing 43 assets of the Hang Seng Market Index (Hong Kong) from 25/11/2005 to 11/04/2016 (daily frequency);
- STOXX50, containing 49 assets of the Euro Stoxx 50 Market Index (Europe) from 22/05/2001 to 11/04/2016 (daily frequency);

- NDX, containing 82 assets of the NASDAQ-100 Market Index (USA) from 03/11/2004 to 11/04/2016 (daily frequency);
- FTSE, containing 83 assets of the FTSE 100 Market Index (UK) from 11/07/2002 to 11/04/2016 (daily frequency).

Stocks with less than ten years of observations were disregarded, thus obtaining a reasonable tradeoff between the number of assets (n) and of observations (T). Furthermore, we have made the daily data sets publicly available on the web site http://host.uniroma3.it/docenti/cesarone/DataSets.htm.

3.2 Computational results

3.2.1 Single Time Window analysis

In Figures 1,2,3,4 we provide the risk-return efficient frontiers $\mathcal{EF}$ for the MV, the SMAD, the CVaR and the MinMax models, respectively, performed on the STOXX50 data set.

In addition, we also report on the risk-return plane the portfolio obtained by (9) (indicated by "X"), and the portfolio computed by solving Problem (8) (indicated by "O") for each utility function listed in Table 1 (see legend of Fig. 1 for an explanation of colors and of symbols).

We observe that the approximation of the maximum Expected Utility (EU) achieved by the portfolio models based on symmetric risk measures (MV and SMAD) tends to be better than that obtained by the portfolio models with asymmetric risk measures (CVaR and MinMax). However, except for MinMax, the approximate solution is very close to the exact one for all the utility functions considered. Furthermore, the behavior highlighted by the computational results is very similar for each data set analyzed, as can be verified by the complete list of the figures made available on the web site http://host.uniroma3.it/docenti/cesarone/papers.htm. In Figs. 5a and 5b we report in the *expected return-expected utility* plane the EU of each portfolio belonging to the efficient frontier (solid line) and the exact EU ("X") for the logarithmic and exponential utility functions with $b = 3$, respectively. However, the behavior for the other utility functions analyzed is similar (see the supplementals available on the web site http://host.uniroma3.it/docenti/cesarone/papers.htm).

Even though the maximum EU obtained from the approximate approach tends to be close to the exact maximum EU, we note that some differences can arise when the risk-aversion is high. In such cases, the MV and the SMAD models work slightly better than the CVaR and MinMax models. The MinMax models seems to generate the worst approximation among all the portfolio selection models analyzed.

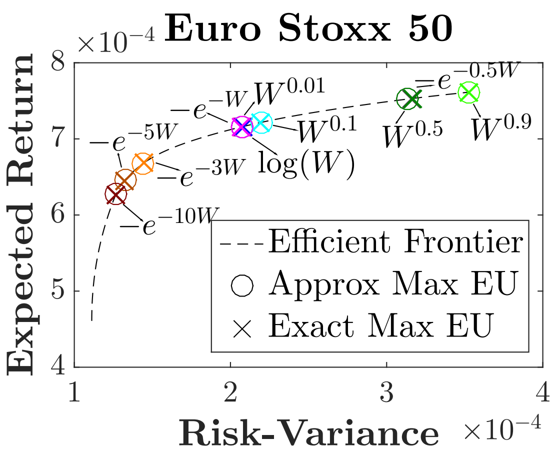


Figure 1: Mean-Variance efficient frontier

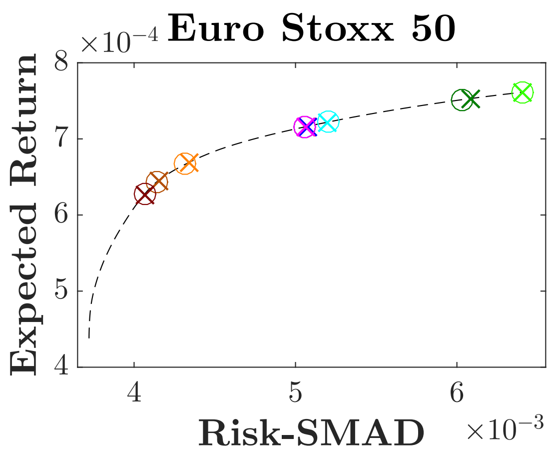


Figure 2: SMAD efficient frontier

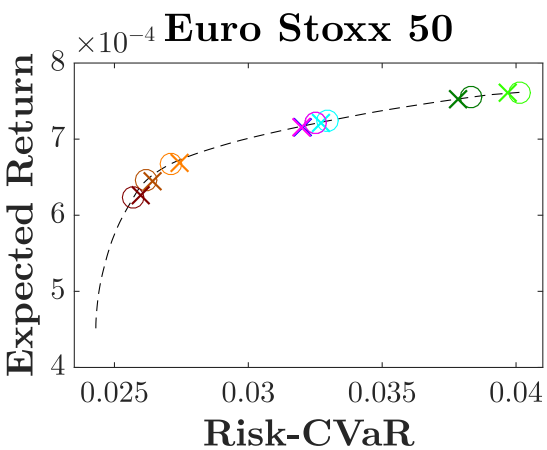


Figure 3: CVaR efficient frontier

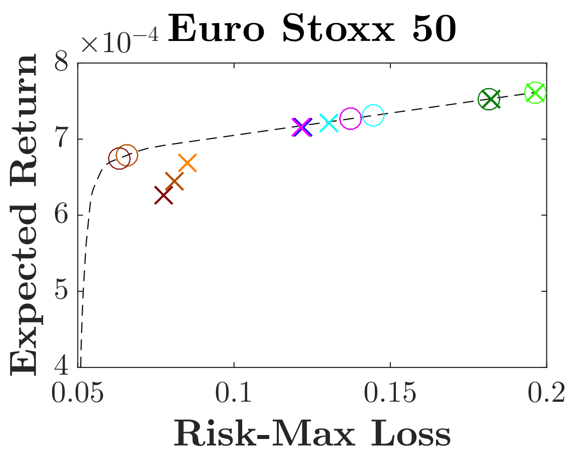
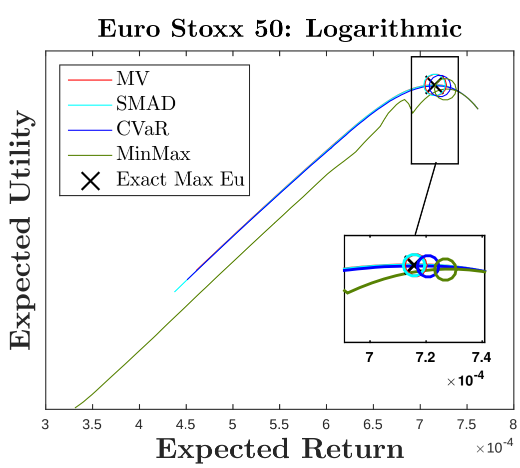
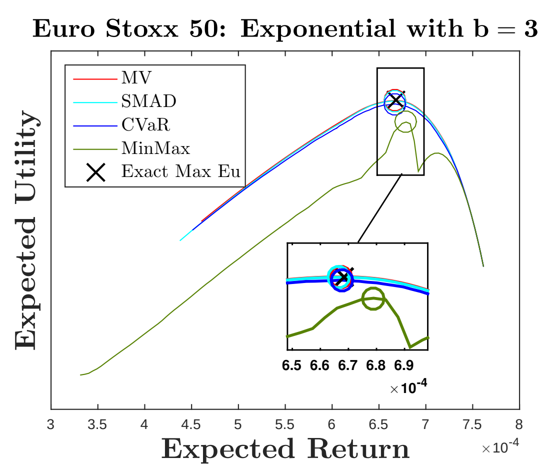


Figure 4: MinMax efficient frontier

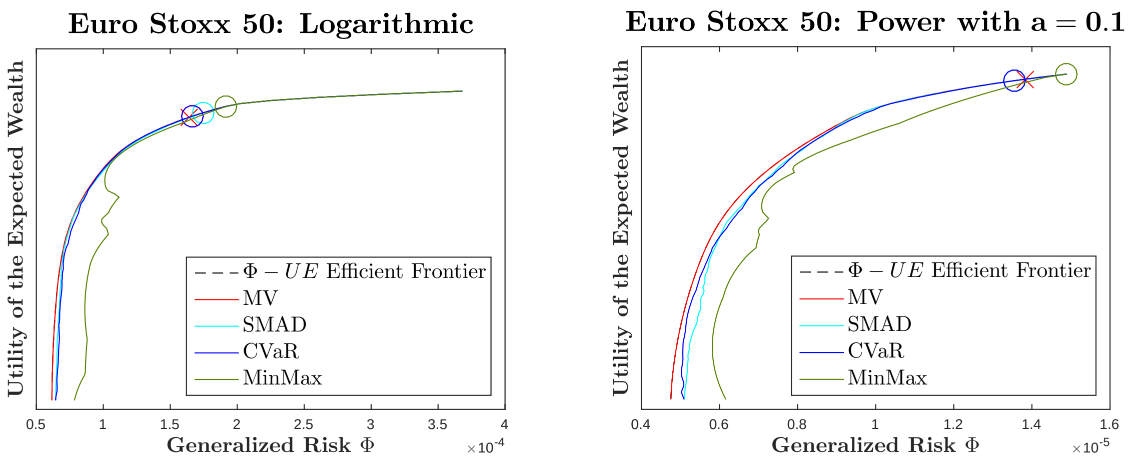


(a) Logarithmic utility



(b) Exponential utility with $b = 3$

Figure 5: Efficient frontiers in the Return-EU plane.



(a) Logarithmic utility

(b) Power utility with $a = 0.1$

Figure 6: Efficient frontiers in the Φ-UE plane.

In Figs. 6a and 6b, we report in the $\Phi - UE$ (Generalized Risk-Utility of the Expected Wealth) plane the efficient frontiers $\mathcal{EF}$ obtained by all the portfolio selection models we considered, along with the exact efficient frontier EF, obtained by solving Problem (2). These results show that the efficient frontiers of the MV and SMAD portfolios tend to coincide with the exact one. The CVaR $\mathcal{EF}$ is very close to the exact one, while that of the MinMax portfolios generally tends to poorly approximate EF.

To the best of our knowledge, this specific analysis seems to be novel for this strand of research.

3.2.2 Rolling Time Windows analysis

To strengthen the results of our empirical analysis, we also adopt a Rolling Time Window scheme of evaluation: we allow for the possibility of rebalancing the portfolio composition during the holding period, at fixed intervals. As in Bruni et al (2015); Cesarone et al (2015, 2016), we use 1000 days (around 4 years) for the in-sample window and 20 for the out-of-sample window, with rebalancing allowed every 20 days (around 1 month). We also refer to Kondor et al (2007) for the choice of the in-sample window length. Indeed, the authors show that for the minimum risk models the estimation error generally tends to be minimal for small values of $\frac{n}{T}$. Furthermore, this choice is especially important for the asymmetric portfolio selection models, as they tend to have estimation errors higher than those of the symmetric ones already for relatively low values of $\frac{n}{T}$. Since in our experimental setup we consider $\varepsilon = 0.05$ (for the CVaR model) and $\frac{n}{T}$ is between 0.028 and 0.083, we expect to have small estimation errors at least for the minimum risk portfolios.

For each risk-return model and for each utility function, in Tables 2, 3, 4 we report how many times the *Approximation Index* (I_{appr}) takes values within different intervals. Since the values assumed by I_{appr} tend to be often close to 1, the intervals

	Logarithmic				Power (a=0.01)				Power (a=0.1)				Power (a=0.5)				Power (a=0.9)			
I_{appr}	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax
$= 1$	**29.2**	5.6	24.3	8.3	**48.6**	23.6	23.6	22.9	**48.6**	23.6	31.3	25.7	**55.6**	52.1	51.4	45.1	79.9	79.2	79.9	**84.0**
[0.99, 1]	98.6	98.6	98.6	78.5	100	100	100	78.5	98.6	98.6	98.6	79.9	97.9	97.9	97.9	93.8	100	100	100	100
[0.95, 0.99)	0	0	0	19.4	0	0	0	19.4	0	0	0	18.8	1.4	1.4	1.4	6.3	0	0	0	0
[0.90, 0.95)	1.4	1.4	1.4	2.1	0	0	0	2.1	0.7	0.7	0.7	1.4	0	0	0	0	0	0	0	0
[0.80, 0.95)	0	0	0	0	0	0	0	0	0	0	0	0	0.7	0.7	0.7	0	0	0	0	0
[0.65, 0.80)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$(-\infty, 0.65)$	0	0	0	0	0	0	0	0.7	0.7	0.7	0	0	0	0	0	0	0	0	0	0
	Exponential (b=0.5)				Exponential (b=1)				Exponential (b=3)				Exponential (b=5)				Exponential (b=10)			
I_{appr}	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax
$= 1$	**38.9**	37.5	34.7	36.1	13.9	12.5	**16.7**	13.2	0	0	**0.7**	0	0.7	0	0	0	**0.7**	0	0	0
[0.99, 1]	100	100	100	92.4	100	100	100	78.5	100	99.3	84.7	22.9	100	97.9	68.1	8.3	100	92.4	42.4	2.1
[0.95, 0.99)	0	0	0	7.6	0	0	0	19.4	0	0.7	15.3	47.9	0	2.1	31.9	29.9	0	7.6	56.9	25.7
[0.90, 0.95)	0	0	0	0	0	0	0	2.1	0	0	0	23.6	0	0	0	29.9	0	0	0.7	16.7
[0.80, 0.95)	0	0	0	0	0	0	0	0	0	0	0	5.6	0	0	0	28.5	0	0	0	23.6
[0.65, 0.80)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3.5	0	0	0	31.3
$(-\infty, 0.65)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.7

Table 2: Observed frequencies (in %) of the Approximation Index values on several representative intervals for the STOXX50 data set.

	Logarithmic				Power ($a = 0.01$)				Power ($a = 0.1$)				Power ($a = 0.5$)				Power ($a = 0.9$)			
I_{appr}	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax
=1	14.7	10.1	**19.4**	4.7	**45.0**	35.7	34.1	32.6	**42.6**	38.0	35.7	34.1	**58.9**	58.9	51.2	57.4	81.4	81.4	**85.3**	82.2
[0.99, 1]	100	96.9	100	79.8	100	96.9	100	80.6	100	98.5	100	83.0	100	100	100	95.4	100	100	100	100
[0.95, 0.99)	0	3.1	0	17.8	0	3.1	0	17.1	0	1.6	0	14.7	0	0	0	4.7	0	0	0	0
[0.90, 0.95)	0	0	0	2.3	0	0	0	2.3	0	0	0	2.3	0	0	0	0	0	0	0	0
[0.80, 0.90)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[0.65, 0.80)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$(-\infty, 0.65)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Exponential ($b = 0.5$)				Exponential ($b = 1$)				Exponential ($b = 3$)				Exponential ($b = 5$)				Exponential ($b = 10$)			
I_{appr}	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax
=1	38.8	41.9	**47.3**	41.1	16.3	16.3	**22.5**	17.1	**0.8**	0	0	0	**0.8**	0	0	0	**2.3**	0	0	0
[0.99, 1]	100	100	100	95.4	100	96.1	100	79.1	100	89.9	86.8	10.1	100	86.8	58.1	2.3	100	71.3	11.6	0
[0.95, 0.99)	0	0	0	4.7	0	3.9	0	17.8	0	10.1	13.2	40.3	0	13.2	41.9	28.7	0	28.7	86.1	3.1
[0.90, 0.95)	0	0	0	0	0	0	0	3.1	0	0	0	29.5	0	0	0	31.8	0	0	2.3	37.2
[0.80, 0.90)	0	0	0	0	0	0	0	0	0	0	0	16.3	0	0	0	31.0	0	0	0	38.8
[0.65, 0.80)	0	0	0	0	0	0	0	0	0	0	0	3.9	0	0	0	6.2	0	0	0	20.9
$(-\infty, 0.65)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 3: Observed frequencies (in %) of the Approximation Index values on several representative intervals for the FTSE data set.

considered are not equally spaced. Indeed, the first two intervals are smaller than the others. Furthermore, note that the first row of these tables represents how frequently the exact solution and the approximate one coincide, i.e., $I_{appr} = 1$. Tables 5, 6, 7 report on average how much is the distance between the approximate and the exact solutions. As described in Section 2.3, the smaller the distance I_{dist}, the better the approximation.

As shown by all the tables, the MV and the SMAD models tend to have better results than those of the CVaR and MinMax models, namely, on average, the models based on symmetric risk measures tend to generate better approximate solutions than those obtained by the models based on downside risk measures. This behavior can be observed both from the averages (on the number of the rolling time windows

	Logarithmic				**Power ($a = 0.01$)**				**Power ($a = 0.1$)**				**Power ($a = 0.5$)**				**Power ($a = 0.9$)**			
I_{appr}	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax
=1	15.2	6.1	**17.2**	7.1	**37.4**	26.3	24.2	29.3	**34.3**	28.3	25.3	25.3	51.5	**54.6**	49.5	53.5	75.8	71.7	**76.8**	72.7
[0.99, 1]	99.0	99.0	100	81.8	100	100	100	82.8	100	100	100	85.9	100	100	100	98.0	100	100	100	100
[0.95, 0.99)	1.0	1.0	0	18.2	0	0	0	17.2	0	0	0	14.1	0	0	0	2.0	0	0	0	0
[0.90, 0.95)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[0.80, 0.90)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
[0.65, 0.80)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$(-\infty, 0.65)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Exponential ($b = 0.5$)				**Exponential ($b = 1$)**				**Exponential ($b = 3$)**				**Exponential ($b = 5$)**				**Exponential ($b = 10$)**			
I_{appr}	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax	MV	SMAD	CVaR	MinMax
=1	**47.5**	46.5	43.4	46.5	**15.2**	14.1	12.1	13.1	**3.0**	0	0	0	0	0	0	0	**4.0**	0	0	0
[0.99, 1]	100	100	100	97.0	100	99.0	100	80.8	100	82.8	59.6	16.2	**100**	69.7	15.2	0	100	42.4	2.0	0
[0.95, 0.99)	0	0	0	3.0	0	1.0	0	19.2	0	17.2	40.4	49.5	0	30.3	80.8	28.3	0	57.6	76.8	0
[0.90, 0.95)	0	0	0	0	0	0	0	0	0	0	0	26.3	0	0	4.0	39.4	0	0	21.2	9.1
[0.80, 0.90)	0	0	0	0	0	0	0	0	0	0	0	8.1	0	0	0	30.3	0	0	0	43.4
[0.65, 0.80)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2.0	0	0	0	36.4
$(-\infty, 0.65)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	11.1

Table 4: Observed frequencies (in %) of the Approximation Index values on several representative intervals for the NDX data set.

		Mean-Variance		**SMAD**		**CVaR**		**MinMax**	
utility function	**Risk Aversion**	**I_{dist} mean**	**I_{appr} mean**	**I_{dist} mean**	**I_{appr} mean**	**I_{dist} mean**	**I_{appr} mean**	**I_{dist} mean**	**I_{appr} mean**
Log		**0.017**	**0.99917**	0.047	0.99836	0.049	0.99845	0.117	0.99374
Power	**$a = 0.01$**	**0.011**	**0.99992**	0.039	0.99912	0.044	0.99923	0.117	0.99384
	$a = 0.1$	**0.018**	**0.99693**	0.044	0.99630	0.043	0.99643	0.110	0.99477
	$a = 0.5$	**0.024**	**0.99866**	0.038	0.99845	0.034	0.99862	0.071	0.99801
	$a = 0.9$	**0.023**	0.99987	0.024	0.99985	0.023	**0.99989**	0.034	0.99968
Exponential	**$b = 0.5$**	**0.018**	**0.99970**	0.029	0.99949	0.026	0.99965	0.077	0.99777
	$b = 1$	**0.015**	**0.99979**	0.042	0.99899	0.045	0.99910	0.119	0.99359
	$b = 3$	**0.009**	**0.99995**	0.056	0.99787	0.092	0.99440	0.232	0.96126
	$b = 5$	**0.009**	**0.99994**	0.056	0.99712	0.102	0.99137	0.287	0.92218
	$b = 10$	**0.013**	**0.99981**	0.058	0.99537	0.112	0.98580	0.316	0.86469

Table 5: Average distance (I_{dist}) between the approximate and the exact solutions and average *Approximation Index* (I_{appr}) for the STOXX50 data set.

examined) of I_{appr} and those of I_{dist} . Furthermore, the MV model almost always seems to achieve the best results, followed by the SMAD. This phenomenon seems to be more evident when the risk-aversion increases.

In Table 8, we report some statistics (mean, median, minimum and maximum) of APL (see Section 2.3) performed on all the in-sample periods of the rolling time window analysis for the STOXX50 data set. The results of this indicator still confirm the behaviour of the risk-return models highlighted by the previous analyses. The efficient frontier $\mathcal{EF}$ obtained by the MV model turns out to be the best approximation for the exact EF, followed by the SMAD model. The models based on the asymmetric risk measures show the worst approximation to the exact efficient frontier.

		Mean-Variance		SMAD		CVaR		MinMax	
utility function	**Risk Aversion**	I_{dist} **mean**	I_{appr} **mean**	I_{dist} **mean**	I_{appr} **mean**	I_{dist} **mean**	I_{appr} **mean**	I_{dist} **mean**	I_{appr} **mean**
Log		**0.020**	**0.99983**	0.040	0.99904	0.042	0.99929	0.130	0.99284
Power	$a = 0.01$	**0.018**	**0.99983**	0.040	0.99906	0.041	0.99928	0.127	0.99304
	$a = 0.1$	**0.021**	**0.99981**	0.038	0.99918	0.039	0.99940	0.108	0.99458
	$a = 0.5$	**0.019**	**0.99980**	0.024	0.99963	0.024	0.99972	0.059	0.99864
	$a = 0.9$	0.022	0.99995	0.023	0.99994	**0.020**	**0.99996**	0.025	0.99995
Exponential	$b = 0.5$	**0.020**	**0.99965**	0.027	0.99946	0.026	0.99956	0.055	0.99843
	$b = 1$	**0.022**	**0.99963**	0.045	0.99877	0.041	0.99901	0.131	0.99246
	$b = 3$	**0.011**	**0.99994**	0.063	0.99680	0.087	0.99519	0.294	0.93564
	$b = 5$	**0.010**	**0.99990**	0.068	0.99500	0.113	0.98997	0.311	0.90947
	$b = 10$	**0.014**	**0.99968**	0.071	0.99117	0.125	0.97893	0.301	0.86480

Table 6: Average distance (I_{dist}) between the approximate and the exact solutions and average *Approximation Index* (I_{appr}) for the FTSE data set.

		Mean-Variance		SMAD		CVaR		MinMax	
utility function	**Risk Aversion**	I_{dist} **mean**	I_{appr} **mean**	I_{dist} **mean**	I_{appr} **mean**	I_{dist} **mean**	I_{appr} **mean**	I_{dist} **mean**	I_{appr} **mean**
Log		**0.016**	**0.99957**	0.043	0.99870	0.033	0.99949	0.074	0.99610
Power	$a = 0.01$	**0.019**	**0.99983**	0.046	0.99897	0.033	0.99948	0.073	0.99629
	$a = 0.1$	**0.018**	**0.99987**	0.039	0.99919	0.031	0.99961	0.069	0.99691
	$a = 0.5$	**0.018**	**0.99985**	0.026	0.99964	0.026	0.99978	0.045	0.99900
	$a = 0.9$	0.023	0.99988	0.023	0.99987	0.025	**0.99990**	**0.023**	0.99989
Exponential	$b = 0.5$	**0.022**	**0.99977**	0.029	0.99955	0.026	0.99970	0.048	0.99888
	$b = 1$	**0.018**	**0.99982**	0.045	0.99889	0.034	0.99941	0.077	0.99612
	$b = 3$	**0.011**	**0.99993**	0.076	0.99452	0.118	0.98928	0.223	0.95600
	$b = 5$	**0.012**	**0.99985**	0.078	0.99180	0.146	0.97547	0.255	0.91655
	$b = 10$	**0.018**	**0.99942**	0.082	0.98594	0.150	0.95961	0.312	0.78803

Table 7: Average distance (I_{dist}) between the approximate and the exact solutions and average *Approximation Index* (I_{appr}) for the NDX data set.

APL		Mean-Variance				SMAD				CVaR				MinMax			
utility function		**mean**	**min**	**max**	**median**	**mean**	**min**	**max**	**median**	**mean**	**min**	**max**	**median**	**mean**	**min**	**max**	**median**
Log		**0.0014**	**0.0001**	**0.0048**	**0.0009**	1.0774	0.2417	4.5419	0.9072	2.7856	0.7826	9.2989	2.5956	24.3638	4.9185	57.8722	21.2625
Power	$a = 0.5$	**0.0665**	**0**	**0.9914**	**0.0008**	1.0801	0.2396	4.5712	0.9147	2.8067	0.7865	9.3458	2.6108	24.4021	4.9281	57.9163	21.3153
Exp	$b = 1$	**0.0005**	**0**	**0.0323**	**0.0002**	1.0833	0.2380	4.6001	0.9194	2.8287	0.7911	9.3953	2.6264	24.4408	4.938	57.9634	21.368
	$b = 3$	**0.0054**	**0.0002**	**0.1906**	**0.0020**	1.0734	0.2456	4.4801	0.8923	2.7456	0.7742	9.2025	2.5597	24.2883	4.8989	57.7811	21.1557
	$b = 5$	**0.0105**	**0.0006**	**0.3168**	**0.0055**	1.0692	0.2534	4.3950	0.8799	2.6640	0.7584	9.0121	2.4841	24.1452	4.8614	57.6032	21.0171
	$b = 10$	**0.0329**	**0.0025**	**0.1172**	**0.0221**	1.0918	0.2484	4.3805	0.8460	2.4742	0.7235	8.5485	2.2501	23.8327	4.7742	57.1782	20.7015

Table 8: Some statistics of APL performed on all the in-sample periods of the rolling time window analysis for the STOXX50 data set.

4 Conclusions

In this paper we addressed the open issue of how the maximum expected utility along the risk-return portfolio efficient frontiers approximates the exact expected utility. We have shown that the approximation of the maximum EU achieved by the portfolio models based on symmetric risk measures (MV and SMAD) tends to be better than that obtained by the portfolio models with asymmetric risk measures (CVaR and MinMax). However, except for MinMax, the approximate solution is very close to the exact one for all the utility functions considered. In addition, in the *Generalized Risk-Utility of the Expected Wealth* plane we have also examined the distance between the exact efficient frontier and the approximate ones. Furthermore, to strengthen the results of our empirical analysis, along with a Single Time Window approach, we have also adopted a Rolling Time Window scheme.

The results of all the indicators, used for evaluating the quality of the approximation, confirm that the efficient frontier obtained by the MV model is almost always the best approximation for the exact efficient frontier, followed by the SMAD model. Conversely, the models based on the asymmetric risk measures show the worst approximation to the exact efficient frontier, in particular that of the MinMax portfolios.

Future research might be directed to find the theoretical foundations of the empirical results obtained here, focusing on the portfolio selection models based on symmetric risk measures.

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